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Schwinger terms in the fully quantized 1 + 1 dimensional model – exact solution

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Abstract. We calculate the equal-time commutator of two fermionic currents within the framework of the $1+1$ dimensional fully quantized theory, describing the interaction of fermions with a vector boson. It is shown that the interaction does not change the result obtained within the theory of free fermions.

1 Introduction

When quantizing the fermionic currents there occur additional terms in the equal-time (E.T.) commutators, so-called Schwinger terms. They can be determined by various methods, for instance perturbatively, see e.g. [1–4], or cohomologically [5,6], and they have a great impact on the theory [1].

Schwinger terms are also closely related to the anomalies of QFT (for an overview see [7]). Whereas anomalies do not get altered by considering *quantized* gauge fields (due to the Adler–Bardeen theorem [8]) this is less clear for the Schwinger terms (ST). Therefore it is our aim to investigate ST in a *fully* quantized theory. We will work in a $1+1$ dimensional QFT describing the interaction of fermions with a vector field. First we consider all fields as massive; then we follow a limiting procedure. In that case all calculations can be performed explicitly and we have a natural continuation of [9], where the case of free fermions was discussed.

The fact that *all* fields are quantized distinguishes this work from others, where similar calculations were done for the theories describing fermions interacting with *external* fields (see e.g. [10–12] and the references given there).

There is one exception (known to the author) [13,14], where the boson field is quantized too, but fermions are considered as *massless* and the procedure used is quite different – from the start the currents are defined as composite operators via the point-splitting method. There is one more difference. The point-splitting method defines a current as some composite operator, whereas in Bogoliubov's definition the current is just one object. The procedure of renormalization reflects this fact because a current is not renormalized as a product of basic fields but rather as an independent object.

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This paper is organized as follows. In Sect. 2 we start with the definition of the "interacting" current $J^\mu_{\rm int}$ [1] introduced by Bogoliubov [15] in the framework of the formalism of Epstein and Glaser [16] which is explained in detail and extensively used in the book by Scharf [17]. Using this definition we derive the explicit form of $J^\mu_{\rm int}$ in the considered two-dimensional field theory model. The calculation of the commutator is done in Sect. 3. The basic properties of free fields are stated in Appendix A. In Appendix B we rigorously show that in the case of our model the formalism of Epstein and Glaser is equivalent to the ordinary one using the $\mathcal{T}$-product (time-ordered product).

2 Definition of the "interacting" current

Following Bogoliubov [15] and Scharf [17] we define

$$J^\mu(x) \equiv \mathcal{S}^{-1}(g)\, \frac{\delta \mathcal{S}(g)}{\mathrm{i}\delta g_\mu(x)}\bigg|_{g_\mu=0}, \tag{2.1}$$

where the S-matrix $\mathcal{S}$ and its inverse $\mathcal{S}^{-1}$ are expressed in perturbative form as

$$\begin{aligned}\mathcal{S}(g) &\equiv \mathbf{1} + \sum_{n=1}^{\infty}\sum_{k=0}^{n} \frac{e^k}{k!\,(n-k)!} \\ &\times \int T_n^{\mu_1,\dots,\mu_{n-k}}(x_1,\dots,x_n) \\ &\times g_{\mu_1}(x_{k+1})\dots g_{\mu_{n-k}}(x_n)\mathrm{d}^2x_1\dots\mathrm{d}^2x_n \quad (2.2)\\ &\equiv \mathbf{1} + T, \quad (2.3)\\ \mathcal{S}^{-1}(g) &\equiv \mathbf{1} + \sum_{n=1}^{\infty}\sum_{k=0}^{n} \frac{e^k}{k!\,(n-k)!}\end{aligned}$$

[1] The label "int" is to emphasize that the operator is *not* a composite operator.

$$\times \int \tilde{T}_n^{\mu_1,\dots,\mu_{n-k}}(x_1,\dots,x_n) \times g_{\mu_1}(x_{k+1})\dots g_{\mu_{n-k}}(x_n)\mathrm{d}^2x_1\dots\mathrm{d}^2x_n, \quad (2.4)$$

where e is the coupling constant of the interaction between fermions and bosons and $g_{\mu_k}(x)$ is a c-number test function from $\mathcal{S}\left(\mathrm{R}^2\right)$ (Schwartz space). The properties of $T_n^{\mu_1,\dots,\mu_{n-k}}(x_1,\dots,x_n)$ are fixed by the required properties of $\mathcal{S}$ (see [17]).

From the equation

$$\mathcal{S}(g)^{-1} = (\mathbf{1}+T)^{-1} = \mathbf{1} + \sum_{r=1}^{\infty}(-T)^r \quad (2.5)$$

we get

$$\tilde{T}_n(X) = \sum_{r=1}^{n}(-)^r \sum_{P_r} T_{n_1}(X_1)\dots T_{n_r}(X_r), \quad (2.6)$$

where the second sum runs over all partitions P_r of $X = \{x_1, x_2, \dots, x_n\}$ into r disjoint non-empty subsets.

For a meaningful definition of our field theory model we have to define the first order terms of (2.2):

$$\mathcal{S}^{(1)}(g) \equiv \int \{eT_1(x) + g_\mu(x)T_1^\mu(x)\}\,\mathrm{d}^2x, \quad (2.7)$$

where

$$T_1(x) = \mathrm{i} : \overline{\psi}(x)\not{A}\psi(x) :, \quad (2.8)$$
$$T_1^\mu(x) = \mathrm{i} : \overline{\psi}(x)\gamma^\mu\psi(x) : . \quad (2.9)$$

The fields $\psi(x)$ and $\overline{\psi}(x)$ represent both fermion and antifermion and the A_μ is a vector boson field. All fields appearing in (2.8) and (2.9) are *free* since we work with a perturbation expansion. The masses of the particles we denote m_ψ for an (anti)fermion and m_A for a boson field. Note that we start with the model containing a massive field in order to avoid problems with infrared singularities. For the further properties of the fields see Appendix A.

Using (2.1) and (2.2) we derive

$$J^\mu(x) = J^\mu_{\mathrm{free}}(x) + \frac{1}{\mathrm{i}}\sum_{n=1}^{\infty}\frac{e^n}{n!}\int A^\mu_{n+1}(x_1,\dots,x_n;x)\mathrm{d}^2x_1\dots\mathrm{d}^2x_n, \quad (2.10)$$

where

$$J^\mu_{\mathrm{free}}(x) =: \overline{\psi}(x)\gamma^\mu\psi(x) : \quad (2.11)$$

and A^μ_{n+1} is the so-called advanced $(n+1)$-point function

$$A^\mu_{n+1}(x_1,\dots,x_n;x) = \sum_{P_2^0}\tilde{T}_m(X\backslash Y)\,T^\mu_{n-m}(Y,x). \quad (2.12)$$

Thus the "interacting" current contains besides the free part also the part which comes from the interaction. The symbol $\sum_{P_2^0}$ denotes the summation over all partitions of the set X including the empty subset $X\backslash Y = \emptyset$. The label "advanced" means that the support of A^μ_{n+1} is

$$\mathrm{supp}\ A^\mu_{n+1}(x_1,x_2,\dots,x_n;x) \subseteq \Gamma^-_{n+1}(x),$$

where

$$\Gamma^-_{n+1}(x) \equiv \left\{\{x_i\}_{i=1}^n \,|\, (x_i - x)^2 \geq 0, x_i^0 \leq x^0\right\},$$

i.e. the A^μ_{n+1} vanish if an arbitrary x_i^0 is greater then x^0.

For reasons which will become clear later we rewrite (2.12) in the form

$$A^\mu_{n+1}(x_1,\dots,x_n;x) = \sum_{\Pi}\theta(x,x_{i_1},\dots,x_{i_n})\,C_n(x,x_{i_1},x_{i_2},\dots,x_{i_n}), \quad (2.13)$$

where

$$C_n(x,x_{i_1},x_{i_2},\dots,x_{i_n}) = [\dots[[T_1{}^\mu(x),T_1(x_{i_1})],T_1(x_{i_2})]\dots,T_1(x_{i_n})] \quad (2.14)$$
$$\theta(x,x_{i_1},\dots,x_{i_n}) = \theta\left(x^0 - x^0_{i_1}\right)\theta\left(x^0_{i_1} - x^0_{i_2}\right)\dots\theta\left(x^0_{i_{n-1}} - x^0_{i_n}\right)$$

and the summation runs over all permutations of the elements of X. (The "advancing" of the support is then evident.)

The step from (2.12) to (2.13) is generally not possible but in our case it can be taken because we work in two dimensions. In this case all functions T_k are well defined (non-singular) and therefore we can multiply $C_n(x,x_{i_1},x_{i_2},\dots,x_{i_n})$ with $\theta(x,x_{i_1},\dots,x_{i_n})$. For further details see Appendix B.

Combining (2.10) and (2.13) we get

$$J^\mu(x) = \frac{1}{\mathrm{i}}\sum_{n=0}^{\infty}e^n J^\mu_n(x), \quad (2.15)$$

where

$$J^\mu_0(x) = \mathrm{i}J^\mu_{\mathrm{free}}(x) = T_1{}^\mu(x), \quad (2.16)$$
$$J^\mu_n(x) = \int\theta(x,x_1,x_2,\dots,x_n) \times C_n(x,x_1,x_2,\dots,x_n)\,\mathrm{d}^2x_1\mathrm{d}^2x_2\dots\mathrm{d}^2x_n. \quad (2.17)$$

Nevertheless the definition (2.1) has to be slightly modified if we (naturally) require that the vacuum expectation value of the "interacting" current $J^\mu(x)$ be equal to zero.

Our redefinition is then straightforward

$$J^\mu_{\mathrm{int}}(x) \equiv J^\mu(x) - \langle 0|\,J^\mu(x)\,|0\rangle \equiv \frac{1}{\mathrm{i}}\sum_{n=0}^{\infty}e^n\left(J^\mu_n(x) - \langle 0|\,J^\mu_n(x)\,|0\rangle\right). \quad (2.18)$$

3 Commutator of "interacting" currents

Now we are ready to calculate the commutator of two "interacting" currents. Using (2.18) we write

$$[J^\mu_{\rm int}(x), J^\nu_{\rm int}(y)]_{\rm E.T.} = [J^\mu(x), J^\nu(y)]_{\rm E.T.} = -\sum_{n=0}^{\infty} e^n \sum_{i=0}^{n} \left[J^\mu_i(x), J^\nu_{n-i}(y)\right], \quad (3.1)$$

and according to formulae (2.17), (2.14) and due to the fact [18] that

$$A^{\mu\nu}_{n+2}(y, x_1, \dots, x_n; x) - A^{\nu\mu}_{n+2}(x, x_1, \dots, x_n; y) = \sum_{i_1 \dots i_n} \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!} \times \left[A^\mu_{k+1}(x_{i_1}, \dots, x_{i_k}; x), A^\nu_{n+1-k}(x_{i_{k+1}}, \dots, x_{i_n}; y)\right], \quad (3.2)$$

we finally get

$$[J^\mu_{\rm int}(x), J^\nu_{\rm int}(y)]_{\rm E.T.} = -\sum_{n=0}^{\infty} e^n \int \theta\,(x, x_1, x_2, \dots, x_n) \times \left[\dots\left[[T_1{}^\mu(x), T_1{}^\nu(y)]_{\rm E.T.}, T_1(x_1)\right]\dots, T_1(x_n)\right] \times {\rm d}^2x_1{\rm d}^2x_2\dots{\rm d}^2x_n. \quad (3.3)$$

However, it turns out that only the first term in the sum contributes. To see this, one has to realize that the *operator relation*

$$[T_1{}^\mu(x), T_1{}^\nu(y)]_{\rm E.T.} \sim \mathbf{1} \quad (3.4)$$

is valid. Indeed, using the Wick theorem we express $[T^\mu_1(x), T^\nu_1(y)]$ in terms of the normally ordered products

$$[T^\mu_1(x), T^\nu_1(y)] = {\rm i} : \overline{\psi}(y)\gamma^\nu S\,(y-x)\,\gamma^\mu\psi(x) : -{\rm i} : \overline{\psi}(x)\gamma^\mu S\,(x-y)\,\gamma^\nu\psi(y) : +{\rm tr}\left\{S^{(-)}(x-y)\,\gamma^\nu S^{(+)}(y-x)\,\gamma^\mu - S^{(-)}(y-x)\,\gamma^\mu S^{(+)}(x-y)\,\gamma^\nu\right\}. \quad (3.5)$$

In the equal-time limit we have

$$S\,(x-y)\,|_{\rm E.T.} = {\rm i}\gamma^0\delta\left(x^1-y^1\right) \quad (3.6)$$

and because in the $1+1$ dimensions the identity

$$\gamma^\mu\gamma^0\gamma^\nu = \gamma^\nu\gamma^0\gamma^\mu \quad (3.7)$$

holds, the relation (3.4) is proved.

Note that here we are working with normal ordered operators, which are well defined in Fock space. Therefore we can indeed write

$${\rm i} : \overline{\psi}(y)\gamma^\nu S\,(y-x)\,\gamma^\mu\psi(x) : -{\rm i} : \overline{\psi}(x)\gamma^\mu S\,(x-y)\,\gamma^\nu\psi(y) : \,|_{\rm E.T.} = 0, \quad (3.8)$$

i.e. the RHS of (3.8) is *not* given as the difference of two infinities [19].

Thus, we can conclude that no contribution from the interaction appears, i.e.

$$[J^\mu_{\rm int}(x), J^\nu_{\rm int}(y)]_{\rm E.T.} = [J^\mu_0(x), J^\nu_0(y)]_{\rm E.T.}\,. \quad (3.9)$$

This is the main result of this article.

Up to now we kept the masses m_ψ and m_A non-zero in order to have all operators well defined. Let us consider a limiting procedure

$$\{m_\psi, m_A\} \to 0,$$

where $\{\dots\} \to 0$ means that the values of one or both elements of $\{\dots\}$ go to zero in the above calculation of the commutator (3.1). It is clear that this procedure exists in our case and is unique in the sense that for arbitrary (non-zero) values of m_ψ and m_A it gives the same result. Moreover, in $1+1$ dimensions it is impossible to consider a massless vector field without introducing an infrared cutoff (or restricting test functions; for details see Appenix A, [20] and the references given there). With this cutoff all operators are well defined and we can conclude that our result is valid in this sense also for the gauge theory. On the other hand, in the case of $m_\psi = 0$ the theory is infrared safe, but for example the calculation of the vacuum polarization is not well defined without introducing an infrared cutoff.

As the author has checked, the same result (3.9) can be obtained in the bosonization scheme [21]. Moreover, it can be shown that [13] (where $m_\psi = 0$) for the model considered here leads, in fact, to the same answer[2]. Therefore a connection between these approaches and our scheme might exist.

4 Conclusion

We have calculated the commutator of "interacting" currents in the simple two-dimensional model describing the interaction of fermions with a vector field. We have shown that the interaction does not change the result obtained within the theory of free fermions. A similar result we also expect to occur in $3+1$ dimensions; however, the corresponding calculations cannot be carried out in the same way in the $3+1$ dimensions because the equality

$$\sum_{P^0_2} \tilde{T}_m\,(X\backslash Y)\,T^\mu_{n-m}\,(Y, x) = \sum_{\Pi} \theta\,(x, x_{i_1}, \dots, x_{i_n})\,C_n\,(x, x_{i_1}, \dots, x_{i_n}) \quad (4.1)$$

is no longer valid.

There is another difference between the $1+1$ and $3+1$ dimensions. The theory in $1+1$ dimensions is finite. This

[2] The result for the commutator of two currents published in [13] is not correct (private communication of the author). Nevertheless for the model considered here the original and corrected results give the same answer.

implies that we do not need any counterterms. Therefore there is no operator mixing in contrast to the $3+1$ dimensions [22] and

$$\left[A^{\mu}_{\text{int}}\left(x\right), J^{\nu}_{\text{int}}\left(y\right)\right]_{\text{E.T.}} = 0.$$

The other types of couplings (e.g. chiral or axial) are under consideration within the framework of our formalism. The results might then be compared (in the limit $m_\psi = 0$) with those of [13].

In view of the intimate connection between the Schwinger terms and the anomaly, our result naturally suggests another (open) question concerning its possible relation to the Adler–Bardeen theorem. However, a *general* proof that quantized gauge fields do not change the result of the external fields is still missing.

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Appendix

A: Fields

For the calculation we have to know the following properties of free fields.

Vector field

The lagrangian of a free massive vector field is

$$\mathcal{L}_A = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m_A^2 A^\mu A_\mu + B\partial_\mu A^\mu + \frac{1}{2}\alpha B^2. \quad \text{(A.1)}$$

The form of the commutator

$$\begin{aligned}\left[A^{(-)}_\mu\left(x\right), A^{(+)}_\nu\left(y\right)\right] &= -\frac{1}{2\pi}\int \mathrm{d}^2p\,\theta\left(p^0\right) \\ &\times \left[g_{\mu\nu}\delta\left(p^2 - m_A^2\right) - \frac{p_\mu p_\nu}{m_A^2}\left(\delta\left(p^2 - m_A^2\right)\right.\right. \\ &\left.\left. - \delta\left(p^2 - \alpha m_A^2\right)\right)\right]\mathrm{e}^{-\mathrm{i}p(x-y)}, \quad \text{(A.2)}\end{aligned}$$

implies its good behavior (as p^{-2}) in the momenta space.

Naively, the limit $m_A^2 \to 0$ leads to the commutator

$$\begin{aligned}\left[A^{(-)}_\mu\left(x\right), A^{(+)}_\nu\left(y\right)\right] &= -\frac{1}{2\pi}g_{\mu\nu} \\ &\times \int \mathrm{d}^2p\theta\left(p^0\right)\delta\left(p^2\right)\mathrm{e}^{-\mathrm{i}p(x-y)}. \quad \text{(A.3)}\end{aligned}$$

Unfortunately, because the distribution

$$\theta\left(p_0\right)\delta\left(p^2\right) \quad \text{(A.4)}$$

is not a well-defined tempered distribution, e.g. if we consider the function $\mathrm{e}_E^{-p^2} = \mathrm{e}^{-\left(p^0\right)^2}\mathrm{e}^{-\left(p^1\right)^2} \in \mathcal{S}$ (Schwartz space) then the integral

$$\int \mathrm{d}^2p\,\theta\left(\pm p_0\right)\delta\left(p^2\right)\mathrm{e}_E^{-p^2} \quad \text{(A.5)}$$

is not finite, and we have two possibilities.

The first one is to say that the massless scalar (vector) field in $1+1$ dimensions does not exist. This deduction is quite legitimate in the framework of the pure QFT.

The other possibility is to violate some of the basic axioms of the QFT as is discussed and done in [20]. We can restrict test functions, or introduce some cutoff term in the definition of the field or redefine $D^{(\pm)}\left(x;0\right)$. We have

$$D^{(\pm)}\left(x;0\right) = \mp\frac{\mathrm{i}}{2\pi}\int \mathrm{d}p^1\frac{1}{|p^1|}\left[\mathrm{e}^{-\mathrm{i}px} - \theta\left(\kappa - |p^1|\right)\right]. \quad \text{(A.6)}$$

Here we use the latter variant (following [20]) but it should be said that all the above-mentioned variants are, as to the final effect, equivalent. Therefore the whole calculation has to be carried out keeping the infrared cutoff finite, and in the chosen approach it is not possible to remove it.

Fermionic field

The Lagrangian of a free fermionic field is

$$\mathcal{L}_f = \overline{\psi}(x)\left(\mathrm{i}\,\partial\!\!\!/ - m_\psi\right)\psi(x). \quad \text{(A.7)}$$

The quantization procedure in $1+1$ dimensions does not contain any infrared problems even in the massless case because of the following form of the commutator:

$$\begin{aligned}\left[\psi^{(-)}_\alpha(x), \overline{\psi}^{(+)}_\beta(y)\right] &= -\frac{\mathrm{i}}{2\pi} \\ &\times \int \mathrm{d}^2p\; p\!\!\!/\,\theta\left(p^0\right)\delta\left(p^2\right)\mathrm{e}^{-\mathrm{i}p(x-y)}, \quad \text{(A.8)}\end{aligned}$$

which behaves as p^{-1} in the momentum space.

B: The transition step

To justify the transition from (2.12) to (2.13) we introduce the distribution D^μ_{n+1}

$$\begin{aligned}D^\mu_{n+1}(x_1,\ldots,x_n;x) \equiv\; & R^\mu_{n+1}(x_1,\ldots,x_n;x) \\ & - A^\mu_{n+1}(x_1,\ldots,x_n;x), \quad \text{(B.1)}\end{aligned}$$

where

$$R^{\mu}_{n+1}(x_1,\ldots,x_n;x) \equiv \sum_{P_2^0} T^{\mu}_{n-m}(Y,x)\,\tilde{T}_m(X\backslash Y)\,. \quad \text{(B.2)}$$

It is possible to show that

$$\text{supp } R^{\mu}_{n+1}(x_1,x_2,\ldots,x_n;x) \subseteq \Gamma^{+}_{n+1}(x)\,, \quad \text{(B.3)}$$

where

$$\Gamma^{+}_{n+1}(x) \equiv \left\{ \{x_i\}^n_{i=1} |\, (x_i - x)^2 \geq 0, x^0_i \geq x^0 \right\} \quad \text{(B.4)}$$

and therefore D^{μ}_{n+1} has a causal support, i.e.

$$\text{supp } D^{\mu}_{n+1}(X,x) \subseteq \Gamma^{+}_{n+1}(x) \cup \Gamma^{-}_{n+1}(x)\,. \quad \text{(B.5)}$$

It is clear that if D^{μ}_{n+1} is not singular then A^{μ}_{n+1} can be expressed as

$$A^{\mu}_{n+1}(x_1,\ldots,x_n;x) = -\prod_{i=1}^{n} \theta\left(x^0 - x^0_i\right) D^{\mu}_{n+1}(x_1,\ldots,x_n;x). \quad \text{(B.6)}$$

Further, we introduce the "truncated" distributions A'^{μ}_{n+1} and R'^{μ}_{n+1}

$$A'^{\mu}_{n+1} \equiv \sum_{P_2} \tilde{T}_m(X\backslash Y)\, T^{\mu}_{n-m}(Y,x)\,, \quad \text{(B.7)}$$

$$R'^{\,\mu}_{n+1} \equiv \sum_{P_2} T^{\mu}_{n-m}(Y,x)\, \tilde{T}_m(X\backslash Y)\,, \quad \text{(B.8)}$$

where $\sum_{P_2}$ means the summation over all partitions of the set X to non-empty subsets. Using (B.7) and (B.8) the distribution (B.1) can be expressed as

$$D^{\mu}_{n+1} = R'^{\mu}_{n+1} - A'^{\mu}_{n+1}. \quad \text{(B.9)}$$

There is one important difference between (B.1) and (B.9). The latter gives us the possibility to express D^{μ}_{n+1} in terms of the n-point function T^n.

Example:

$$D^{\mu}_2(x_1;x) = R'^{\mu}_2(x_1;x) - A'^{\mu}_2(x_1;x) \quad \text{(B.10)}$$

and if D^{μ}_2 is not singular then we can write

$$A^{\mu}_2(x_1;x) = -\theta\left(x^0 - x^0_1\right) D^{\mu}_2(x_1;x), \quad \text{(B.11)}$$

i.e. we split D^{μ}_2. Then using (B.11) we get

$$A^{\mu}_2(x_1;x) = -\theta\left(x^0 - x^0_{\pi_1}\right)\left(R'^{\,\mu}_2(x_1;x) - A'^{\,\mu}_2(x_1;x)\right) = \theta\left(x^0 - x^0_{\pi_1}\right)\left[T^{\mu}_1(x), T_1(x_1)\right]. \quad \text{(B.12)}$$

Furthermore according to the definitions (2.12), (B.7), (B.2) and (B.8) we have

$$T^{\mu}_n(x_1,\ldots,x_{n-1},x) = R^{\mu}_n - R'^{\mu}_n = A^{\mu}_n - A'^{\mu}_n \quad \text{(B.13)}$$

and repeatedly using (B.6) we can finally express D^{μ}_{n+1} in terms of the 1-point function $T^{(\mu)}_1$. In that way we get A^{μ}_{n+1} in (2.13) by combining (B.6) and the above procedure.

This routine is equivalent [18] to the application of the following equality:

$$T^{\mu}_{n+1}(x_1,\ldots,x_n,x) = \mathcal{T}\left(T^{\mu}_1(x)T_1(x_1)\ldots T_1(x_n)\right),$$

where

$$\mathcal{T}\left(T^{\mu}_1(x)T_1(x_1)\ldots T_1(x_n)\right) = \sum_{\Pi} \theta\left(x, x_{i_1},\ldots,x_{i_n}\right) T^{\mu}_1(x)T_1(x_{i_1})\ldots T_1(x_{i_n}). \quad \text{(B.14)}$$

Example:

$$\begin{aligned} T^{\mu}_2(x_1,x) &= -\theta\left(x^0 - x^0_1\right) D^{\mu}_2 - A'^{\mu}_2 \\ &= -\theta\left(x^0 - x^0_1\right) R'^{\mu}_2 - \left(1 - \theta\left(x^0 - x^0_1\right)\right) A'^{\mu}_2 \\ &= \theta\left(x^0 - x^0_1\right) T^{\mu}_1(x)T_1(x_1) + \theta\left(x^0_1 - x^0\right) T^{\mu}_1(x)T_1(x_1) \\ &= T\left(T^{\mu}_1(x)T(x_1)\right). \end{aligned} \quad \text{(B.15)}$$

However, as was shown in [17] the splitting of an arbitrary distribution with causal support to a retarded and an advanced part via multiplication by the combination of theta functions, i.e. (B.6), is not generally a well-defined procedure.

The reason why we can do it here is that we work in two dimensions. We show that the terms $T_{i_1}\ldots T_{i_k}, i_j \in \{1,\ldots,n\}$, $\sum_{j=1}^k i_j = n+1$, which are "sitting" in D_{n+1}, are correctly defined and they have a non-singular behavior. The last property enables their multiplication by the combination of the theta functions.

Every term $T_{i_1}\ldots T_{i_k}$ is expressible as a sum of terms of the normally ordered operators (graphs) of the form

$$T_{i_1}\ldots T_{i_k} \sim \sum_k T^{g_k}_{n+1}(x_1,\ldots,x_n,x) \quad \text{(B.16)}$$

where

$$T^g_n(x_1,\ldots,x_n) =: \prod_{i=1}^{n_f} \overline{\psi}(x_{k_j}) t_g(x_1,\ldots,x_n) \prod_{i=1}^{n_f} \psi(x_{n_j}) :: \prod_{i=1}^{n_b} A\left(x_{m_j}\right) : \quad \text{(B.17)}$$

and n_f is the number of external fermions (or antifermions), n_b the number of the external massive bosons and $t_g(x_{1,}\ldots,x_n)$ is c-number distribution.

In the dimension d the graph g (B.17) has the singular order

$$\omega(g) = n\left(\frac{d}{2} - 2\right) + d - n_b\left(\frac{d}{2} - 1\right) - n_f(d-1)\,, \quad \text{(B.18)}$$

and for $d = 2$ we get

$$\omega(g) = 2 - n - n_f. \quad \text{(B.19)}$$

We see that the problematic (singular) case $\omega(g) \geq 0$ can appear only for $n = 2$. All higher-order graphs do not contain any singularity. Moreover the c-number distribution in the 2-point causal function really has $\omega(g) = -2$. This all means that all graphs (including their subgraphs) are not singular, the terms $T_{i_1} \dots T_{i_k}$ are well defined and can be multiplied by the combination of theta functions.

Therefore the formula (2.13) is consistent with (2.12).

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